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On a certain class of cyclically presented groups

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ABSTRACT

We continue the study of the cyclically presented groups $G = G_n(w)$ where $w = x_0 x_i^{-\alpha} x_j^{-\beta} x_i^{\alpha} x_j^{\beta}$. We prove that under certain conditions on the parameters n, i, j, α, β the group G is infinite. These results support our conjecture which states for $n > 4$ exactly when G is trivial and that if G is non-trivial then G is infinite.

1. Introduction

Let $F_n = \langle x_0, \ldots, x_{n-1} \mid \, \rangle$ be the free group of rank n and let ϕ be the automorphism of F_n determined by the correspondences $x_i\phi = x_{i+1}$, where subscripts are taken modulo n. Let w be a cyclically reduced word in the generators x_i and let N be the normal closure in F_n of $\{w, w\phi, \ldots, w\phi^{n-1}\}$. Define $G_n(w) = F_n/N = \langle x_0, \ldots, x_{n-1} \mid w, w\phi, \ldots, w\phi^{n-1}\rangle$. A group G is said to have a *cyclic presentation* or to be *cyclically presented* if $G \cong G_n(w)$ for some w and for some n. Moreover the cyclic presentation for $G_n(w)$ is said to be *irreducible* either when $n = 1$ or when $n \geqslant 2$ and the following two conditions hold: w involves at least two of the x_i; if w involves only $x_{i_1}, \ldots, x_{i_k}$ where $i_j < i_{j+1}$ $(1 \leqslant j < k)$ and where $k \geqslant 2$ then $\mathrm{hcf}(i_2 - i_1, \ldots, i_k - i_{k-1}, n) = 1$.

In this paper we consider $G_n(w)$ for $w = x_0[x_i^{\alpha}, x_j^{\beta}]$ where $\alpha, \beta \in \mathbb{Z} \setminus \{0\}$, $[a, b]$ denotes $a^{-1}b^{-1}ab$ and $0 \leqslant i, j < n$, $i \neq j$, with the understanding that if $i = 0$ then $\alpha < 0$ or if $j = 0$ then $\beta > 0$. Early examples appear in papers of Higman [6] and Neumann [10] where it is shown that $G_n(x_1^{-1}x_0x_1x_0^{-2})$ $(= G_n(x_0[x_0^{-1}, x_1]))$ is trivial for $n = 2, 3$ and infinite for $n > 3$. It follows that $G_2(x_0[x_0^{-k}, x_1^{\beta}])$ and $G_3(x_0[x_0^{-1}, x_1^{\beta}])$ are trivial for $k \geqslant 2$, however $G_3(x_0[x_0^{-k}, x_1])$ was proved infinite for $k \geqslant 2$ by Neumann [11]. Further irreducible, trivial examples are $G_3(x_0[x_1, x_2])$, $G_n(x_0[x_1^{-1}, x_2^{-1}])$ for $3 \leqslant n \leqslant 4$ and $G_4(x_0[x_1, x_3])$ which appeared in [4] and this last example was generalised by Havas and Robertson [5] who noted that if $n = 2k \geqslant 4$ and $\alpha \in \mathbb{Z} \setminus \{0\}$ then $G_n(x_0[x_i^{\alpha}, x_{i+k}^{\alpha}])$ is irreducible and

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trivial whenever $\text{hcf}(i,k) = 1$. Finally in [3] it is shown that $G_n(x_0[x_1^\alpha, x_2^\beta])$ is infinite for $n \geqslant 5$ and $(|\alpha|, |\beta|) \neq (1,1)$.

The main results proved here are the following.

Theorem 1.1. *Let $G = G_n(x_0[x_i^\alpha, x_j^\beta])$ where $\alpha, \beta \in \mathbb{Z} \setminus \{0\}$ and at least one of $\text{hcf}(n,i)$, $\text{hcf}(n,j)$ equals 1. If $n = 2^k m$ where $k \geqslant 0$, $m \geqslant 11$ is odd, $|\alpha| > 1$, $|\beta| > 1$, $|\alpha| \neq |\beta|$ and $n \neq 2(j-i)$ then G is infinite.*

If $\text{hcf}(n,i) = 1$ or $\text{hcf}(n,j) = 1$ then $\text{hcf}(n,i,j) = 1$ which is equivalent to the presentation for G being irreducible. The proof of Theorem 1.1 will take up most of this paper.

Elsewhere [13] the second-named author has shown that $G_6(x_0[x_2^\alpha, x_3^\beta])$ is infinite for $|\alpha| > 1$ and $|\beta| > 1$. As a consequence of this we can give a short proof in the final section of the following.

Theorem 1.2. *Let $G = G_n(x_0[x_i^\alpha, x_j^\beta])$ where $\alpha, \beta \in \mathbb{Z} \setminus \{0\}$, $\text{hcf}(n,i,j) = 1$, $\text{hcf}(n,i) > 1$ and $\text{hcf}(n,j) > 1$. If $|\alpha| > 1$ and $|\beta| > 1$ then G is infinite.*

There is interest in balanced presentations that define the trivial group. For the groups considered here we believe that for $n \geqslant 5$ the only trivial examples are of Havas–Robertson type.

Conjecture 1. *Let $G = G_n(x_0[x_i^\alpha, x_j^\beta])$, where $n \geqslant 5$, be irreducible. Then G is trivial if and only if $n = 2k$, $j = i + k \pmod n$, $\alpha = \beta$ and $\text{hcf}(i,k) = 1$. Moreover, if G is not trivial then G is infinite.*

Our theorems can be seen as a step toward proving this conjecture. (Note that the conditions of the conjecture imply either $\text{hcf}(n,i) = 1$ or $\text{hcf}(n,j) = 1$ and that $n = 2(j-i)$.) For small values of the parameters there is computational evidence for the conjecture to be true; more precisely a computation with KBMAG [7] shows that for $5 \leqslant n \leqslant 10$ and $0 < |\alpha|, |\beta| \leqslant 3$ the conjecture holds for all except one non-trivial case whose order is still to be determined. (Note however that the trivial-infinite dichotomy does not hold for $n < 5$. For example $G_3(x_0[x_1^{-1}, x_2])$ has order 120.)

The proofs make use of curvature arguments. The method and preliminary results are given in Sections 2 and 3 from which we obtain the following.

Corollary 1.3. *Let $G = G_n(x_0[x_i^\alpha, x_j^\beta])$ where $0 < i < j < n$ be irreducible and assume that $|\alpha| > 1$, $|\beta| > 1$ and $|\alpha| \neq |\beta|$. If $j < \frac{n}{3}$ or $j - i > \frac{2n}{3}$ then G is infinite.*

The proofs of Theorems 1.1 and 1.2 are given in Section 4.

2. Method of proof

For a fixed n, we will work *modulo* an equivalence relation on the set of presentations $G_n(x_0[x_i^\alpha, x_j^\beta])$. Two words $w = x_0[x_i^\alpha, x_j^\beta]$ and $w' = x_0[x_{i'}^{\alpha'}, x_{j'}^{\beta'}]$ are said to be *equivalent*, and we write $w \sim w'$, if w' can be obtained from w by a sequence of the following *elementary moves*:

(E1) replacing x_k with x_k^{-1} for each k;
(E2) cyclic permutation;
(E3) inversion;
(E4) permutation of subscripts induced by an automorphism of $\mathbb{Z}_n$.

Clearly $\sim$ is an equivalence relation and since each elementary move does not change the isomorphism class it follows that if $w \sim w'$ then $G_n(w) \cong G_n(w')$. We note that it follows from Theorem 3 in [12] that if $n \geqslant 4$, $G = G_n(x_0[x_i^\alpha, x_j^\beta])$ is irreducible and either $i = 0$ or $j = 0$ then G is infinite. In view of this we can assume for $n \geqslant 4$, using elementary moves if necessary, that $0 < i < j < n$

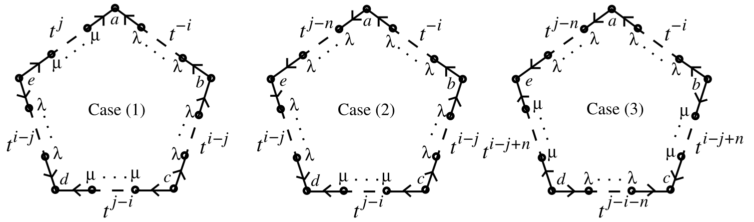

Fig. 1. Labels of m-regions of $\mathscr{D}$.

Table 1
Values of labels in $\mathscr{D}$.

a	b	c	d	e	s	λ	μ
1	$-\alpha$	$-\beta$	α	β	0	0	0

and $i \leqslant \frac{n}{2}$. Subject to these constraints $(\alpha', \beta') \in \{(\alpha, \beta), (-\beta, -\alpha)\}$ therefore any condition we assume on $|\alpha|$ must also be assumed for $|\beta|$. Note further that elementary moves take irreducible cyclic presentations to irreducible cyclic presentations.

The automorphism ϕ of F_n described in the introduction induces an automorphism of G using which we can form the split extension $E = E(n; \alpha, \beta; i, j)$ of G by the cyclic group of order n. Thus $E = \langle x, t \mid t^n, xt^{-i}x^{-\alpha}t^{i-j}x^{-\beta}t^{j-i}x^{\alpha}t^{i-j}x^{\beta}t^{j}\rangle$ (see, for example, [8, Chapter 6]) and in order to show that G is infinite it is enough therefore to prove that E is infinite. Since t has order n in the extension E it follows that we can ensure that all the t-exponents in the second relator are in the interval $[-\frac{n}{2}, \frac{n}{2}]$; since we are assuming $0 < i < j < n$ and $i \leqslant \frac{n}{2}$, we have the following three cases:

(1) $j \leqslant \frac{n}{2}$, in which case all the t-exponents are already in the given interval and they are *reduced modulo n*;
(2) $j > \frac{n}{2}$ and $j - i \leqslant \frac{n}{2}$, in which case j is not reduced modulo n and becomes $j - n$ after reduction;
(3) $j - i > \frac{n}{2}$, in which case j, $j - i$ and $i - j$ reduce to $j - n$, $j - i - n$ and $i - j + n$ respectively.

We prove E infinite by showing that x has infinite order in E. Assume, by way of contradiction, that $|x| < \infty$. Then (see, for example [1] or [9]) there exists a van Kampen diagram K over $E(n; \alpha, \beta; i, j)$ whose boundary is a simple closed curved with label x^l for some $0 < l < \infty$. Over all such diagrams we will assume K to be minimal with respect to the number of regions. Now collapse to a point each edge in K which is labelled by $x^{\pm 1}$. What we obtain is a so-called *modified van Kampen diagram* which is a tessellation $\mathscr{D}$ of the 2-sphere whose regions inherit the labelling from the regions of K. Regions corresponding to the relator t^n are left unchanged by the collapses and are called *s-regions* and their corners are labelled by s, while regions corresponding to the other relator are called *m-regions* and are given (up to inversion) by Fig. 1. Here we use powers of t to label sequences of edges and the corner labels correspond to the powers of x appearing in the second relator, so their values are given by Table 1. We will often overline a label to denote its inverse, except for λ and μ which are mutually inverses zero-valued labels.

The vertex obtained from collapsing the boundary of K is called the *distinguished vertex* and will be denoted by v_0; vertices which are not distinguished are said to be *interior* and are usually denoted by v_r where r is the label of the corresponding corner in the considered region. A region having v_0 as a vertex is called a *boundary region*, otherwise a region is said to be *interior*. Two regions are said to be *adjacent* if they share an edge. The *label* $l(v)$ of a vertex v in $\mathscr{D}$ is the word given by the corner labels at v read anti-clockwise (and so is defined up to cyclic permutation); the *label sum* of $l(v)$ is the sum of the values of the corner labels at v. It follows that the label sum of each interior vertex will be 0 and that the label sum of the distinguished vertex will be l. When we need to write multiple labels we will often use brackets to list the possible choices (for example $a\{\lambda, s\}b$ means $a\lambda b$ or asb).

Fig. 2. Star graphs in cases (1) and (3).

In cases (1) and (3) the associated *star graph* Γ of $\mathscr{D}$ is given in Fig. 2 (i) and (ii) respectively, with the understanding that each edge can be traversed in either of the two directions except for those labelled by λ and μ which can be traversed only according to their orientation (in fact they are inverses of each other). A path in Γ is said to be *admissible* if it is closed, cyclically reduced and has 0 as sum of its edge labels. The label of an interior vertex in $\mathscr{D}$ corresponds to an admissible path in Γ. This gives us a way to construct a list of all possible labels (and their consequences for α and β, if any) for a vertex of a given degree.

We will use the same curvature argument as in [3]. The *degree* of a vertex v in $\mathscr{D}$, denoted by $d(v)$, is the number of edges occurring at v. Each corner of $\mathscr{D}$ is given an angle θ and the *curvature* of a vertex v, denoted by $c(v)$, is equal to $2\pi - \sum_{i\in I}\theta_i$, where I is a set of indices such that the θ_i are the angles occurring at v. Define the *degree* $d(\Delta)$ of a region Δ of $\mathscr{D}$ to be the number of vertices in Δ of degree $\geqslant 3$ (so we do not count the vertices of degree 2). The *curvature* $c(\Delta)$ of a region Δ of $\mathscr{D}$ with k edges and whose angles are $\theta_1,\dots,\theta_k$ is defined to be $(2-k)\pi + \sum_{i=1}^{k}\theta_i$. To each corner at a vertex v we give an angle $\frac{2\pi}{d(v)}$. It is a consequence of Euler's formula that the total curvature T of $\mathscr{D}$, defined to be the sum of curvatures of all vertices and regions of $\mathscr{D}$, equals 4π. If Δ is a region with vertices $v_1,\dots,v_k$ we write $c(\Delta) = c(d_1,\dots,d_k)$ where $d_i = d(v_i)$. It follows that $c(v) = 0$ for each vertex of $\mathscr{D}$ and $c(2,\dots,2,d_1,\dots,d_l) = c(d_1,\dots,d_l) = (2-l)\pi + \sum_{i=1}^{l}\frac{2\pi}{d_i}$ so we can ignore vertices of degree 2 when evaluating $c(\Delta)$.

In order to obtain the desired contradiction we introduce the *pseudo-curvature* $c^*(\Delta)$ of a region Δ which is given by $c(\Delta)$ plus all the positive curvature Δ receives and minus all the positive curvature transferred from Δ according to a distribution process which will be specified later. We then proceed, case by case, as follows:

i) we classify the positively curved interior regions and describe a distribution process (or compensation scheme) for their curvature;
ii) we show that each interior region has non-positive pseudo-curvature;
iii) we then show that the pseudo-curvature of each boundary region is strictly less than $\frac{4\pi}{k_0}$ (where k_0 is the degree of the distinguished vertex).

Since $T = \sum_{\Delta\in\mathscr{D}} c(\Delta) = \sum_{\Delta\in\mathscr{D}} c^*(\Delta)$ and since there are at exactly k_0 boundary regions it follows from ii) and iii) that $T < 4\pi$ and so E is infinite. The first step is to classify the positively curved interior regions of $\mathscr{D}$; it follows from the definition of curvature that if $d(\Delta) \geqslant 6$ then $c(\Delta) \leqslant 0$ so we need consider only the interior regions of degree $\leqslant 5$.

We introduce some terminology. An *m-segment* of $\mathscr{D}$ is a sequence of t-edges of an m-region whose end corners are labelled by a non-zero power of x and the other corners are labelled by zero-valued labels (in other words it is a sequence of t-edges corresponding to the powers of t in the second relator of the extension after reduction modulo n). An *s-segment* is a sequence of edges of an s-region whose vertices have degree 2 except for the end vertices which have degree > 2. A *splitting* in a segment of an m-region is a vertex of degree > 2 in the segment and which is not an end vertex. It follows from the definition that there are five m-segments in each m-region. The m-segment whose end vertices are v_y and v_z will be denoted by yz and we will use $|yz|$ to denote the *length* of yz which is the number of edges in yz. Thus $(y,z) \in \{(a,b),(b,c),(c,d),(d,e),(e,a)\}$ up to inversion. If there is a splitting of degree k in some m-segment we say that the m-segment *splits in degree k*. Moreover we use the term *segment* to refer to both m-segments and s-segments when there is no confusion.

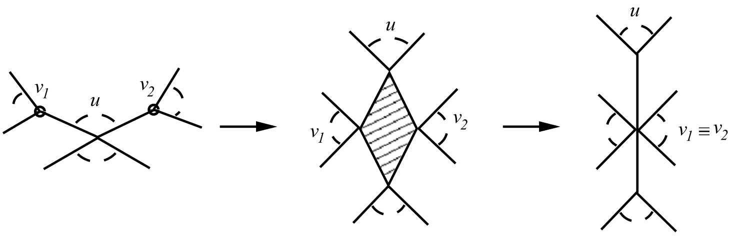

Fig. 3. Bridge move relative to the sublabel u.

3. Preliminary results

In this section we prove results required for the proof of Theorem 1.1 and so it will be assumed throughout that $|\alpha| > 1$, $|\beta| > 1$ and $|\alpha| \neq |\beta|$. In order to find some conditions on i and j under which G is infinite we shall consider a modified van Kampen diagram corresponding to $|x| < \infty$ in E. We can assume without any loss of generality that each of the following conditions holds:

A1 $\mathscr{D}$ is minimal with respect to the number of regions.
A2 Subject to **A1**, the number of vertices in $\mathscr{D}$ with label ce (up to inversion and cyclic permutation) is maximal.
A2* When $j - i > \frac{n}{2}$ and subject to **A2**, the number of vertices in $\mathscr{D}$ with label bd (up to inversion and cyclic permutation) is maximal.
A3 Subject to **A2** (and to **A2*** when $j - i > \frac{n}{2}$), the number of vertices of $\mathscr{D}$ of degree 2 is maximal.
A4 When $j - i > \frac{n}{2}$ each s-region in $\mathscr{D}$ has degree $\geqslant 4$.

Using *bridge moves* (see Fig. 3 and, for example, [2]) we can prove the following.

Lemma 3.1. *If $G_n(x_0[x_i^{\alpha}, x_j^{\beta}])$ is in case* (1) *or* (3)*, $\mathscr{D}$ satisfies* **A1**, **A2** (*and* **A2*** *in case* (3)) *and* **A3** *above and $l(v)$ is a vertex label, then the following hold*:

(i) *$l(v)$ is cyclically reduced*;
(ii) *$l(v)$ cannot have as proper sublabel $xw\bar{x}$ or $\bar{x}wx$ where w is a sublabel with zero label sum and $x \in \{a, b, c, d, e\}$*;
(iii) *ce, up to cyclic permutation and inversion, cannot appear as a proper sublabel of $l(v)$*;
(iv) *in case* (1) *we cannot have $l(v) = y_1y_2y_3y_4$, up to cyclic permutation and inversion, where y_i is a zero-valued label for $i = 1, 2, 3$ and* 4;
(v) *in case* (3) *we cannot have $l(v) = sxsy$, up to cyclic permutation and inversion, where x and y are zero-valued labels*;
(vi) *in case* (3) *bd, up to cyclic permutation and inversion, cannot appear as a proper sublabel of $l(v)$.*

Proof. If (i) or (ii) does not hold we can reduce the number of regions contradicting **A1**. If (iii) does not hold we can apply a bridge move at v relative to the sublabel ce creating a new vertex with label ce hence contradicting **A2**. If (iv) or (v) does not hold we can apply bridge moves at v creating two new vertices of degree 2 and killing at most one vertex of degree 2, hence contradicting **A3** (here in order to get the contradiction we might need to apply repeatedly the same bridge move until a vertex of degree > 2 will eventually be involved). It remains to prove (vi). Suppose $\mathscr{D}$ satisfies **A1**, **A2**, **A2*** and **A3** and bd is a subword of $l(v)$. A bridge move at v relative to bd creates a new vertex with label bd; moreover it does not change the number of regions and does not kill any vertex labelled by ce (up to cyclic permutation and inversion) since in case (3) $|de| = n - j + i > 1$. It follows that the new diagram satisfies **A1**–**A2** but contradicts **A2***. $\square$

The list of possible labels for vertices of degree 2 and 3 involving a, b, c, d, e or s (up to cyclic permutation and inversion) is as follows.

$$\begin{array}{rll}
\text{For } j \leqslant \frac{n}{2}\text{: degree 2:} & ce, s\lambda & \text{with no consequence for } \alpha \text{ and } \beta; \\
\text{degree 3:} & bds, bd\mu & \text{with no consequence for } \alpha \text{ and } \beta; \\
& daa \Rightarrow \alpha = -2;\ \bar{d}aa \Rightarrow \alpha = 2. & \\
\text{For } j - i > \frac{n}{2}\text{: degree 2:} & ce, bd, s\lambda & \text{with no consequence for } \alpha \text{ and } \beta; \\
\text{degree 3:} & \text{none.} &
\end{array}$$

Now we classify the positively curved interior regions of $\mathscr{D}$.

Lemma 3.2. *Let Δ be an interior s-region in $\mathscr{D}$. Then the following hold*:

(i) *if $j < \frac{n}{3}$ then $c(\Delta) \leqslant 0$;*
(ii) *if $j - i > \frac{n}{2}$ and $d(\Delta) \geqslant 4$ then $c(\Delta) \leqslant 0$.*

Proof. If $d(\Delta) \geqslant 6$ the result is immediate in both cases. Furthermore for $j < \frac{n}{3}$, each s-region of $\mathscr{D}$ has degree $\geqslant 4$ since in this case j is the maximal m-segment length. Therefore assume that $4 \leqslant d(\Delta) \leqslant 5$. First assume $j < \frac{n}{3}$. Suppose v_1 and v_2 are two vertices of Δ of degree 3, so $l(v_1) = l(v_2) = bds$. Since db is not a segment label it follows that there must be a vertex of degree $\geqslant 4$ between v_1 and v_2. It follows that if $d(\Delta) = 5$ then Δ cannot have more than two vertices of degree 3, therefore $c(\Delta) \leqslant c(3,3,4,4,4) = -\frac{\pi}{6}$. Now let $d(\Delta) = 4$. If Δ has a vertex of degree 3 then it is labelled by bds, therefore Δ has a segment of length $\leqslant |ab| = i$ and a segment of length $\leqslant |de| = j - i$. Since the maximal m-segment length is $j < \frac{n}{3}$ it follows that the sum of segment lengths of Δ is strictly less than n, a contradiction. It follows that there is no vertex of degree 3 in Δ, hence $c(\Delta) \leqslant c(4,4,4,4) = 0$.

Now suppose $j - i > \frac{n}{2}$ and let Δ be an s-region of degree $\geqslant 4$; since an interior vertex cannot have degree 3, it follows that $c(\Delta) \leqslant c(4,4,4,4) = 0$. $\square$

Lemma 3.3. *Let Δ be an interior m-region in $\mathscr{D}$. If $j < \frac{n}{3}$ then $c(\Delta) \leqslant 0$.*

Proof. According to the geometric construction of the modified van Kampen diagram for $j \leqslant \frac{n}{2}$ the lengths of segments in Δ are $|ab| = i$, $|ea| = j$ and $|bc| = |cd| = |de| = j - i$.

First assume $d(v_c) = d(v_e) = 2$. Since $|bc| = j - i < j = |ea|$ the segment ea splits and the splitting has degree $\geqslant 4$ since it has sublabel $\{b, \lambda\}\mu$. If there is a splitting in the segment bc then this splitting must have sublabel $\lambda\mu$ (because $|ea| > |bc|$), hence degree $\geqslant 4$, and it follows that $c(\Delta) \leqslant c(3,3,3,4,4) = 0$. We can therefore assume that the segment bc does not split. Since $|ea| > |bc|$ it follows that v_b has sublabel $b\mu$, assume therefore $d(v_b) \geqslant 4$. If there is another splitting $c(\Delta) \leqslant c(3,3,3,4,4) = 0$ so assume otherwise. Since $|cd| = |de|$, it follows that v_d has sublabel ddd so that $d(v_d) \geqslant 6$ and $c(\Delta) \leqslant c(3,4,4,6) = 0$.

Now assume that only one of the vertices v_c and v_e has degree 2.

If $d(v_e) = 2$ and $d(v_c) > 2$ as before the segment ea splits in degree $\geqslant 4$; moreover $d(v_c) \geqslant 4$ and $d(v_a), d(v_b), d(v_d) \geqslant 3$, therefore $c(\Delta) \leqslant c(3,3,3,4,4) = 0$.

Now let $d(v_c) = 2$ and $d(v_e) > 2$. If one of the segments bc or cd splits then the splitting has degree $\geqslant 4$ because $|ea| > |bc|$ and $|de| = |cd|$, respectively. So we can assume there is no splitting (otherwise $c(\Delta) \leqslant 0$) and therefore v_d has sublabel dd and v_b has sublabel $b\mu$; hence $d(v_d) \geqslant 5$ and $d(v_b) \geqslant 4$. If $d(v_a) \geqslant 4$ then $c(\Delta) \leqslant c(4,4,4,5) < 0$; if $d(v_a) = 3$ then $l(v_a) \in \{aad, aa\bar{d}\}$ and comparing the segments lengths it turns out that ea must split. It follows that $c(\Delta) \leqslant c(3,3,4,4,5) < 0$.

Finally, if $d(v_e), d(v_c) > 2$ then $c(\Delta) \leqslant c(3,3,3,4,4) = 0$. $\square$

Lemma 3.4. *Let Δ be an interior m-region in $\mathscr{D}$. If $j-i>\frac{n}{2}$ and $c(\Delta)>0$ then Δ is one of the regions in Fig. 4, hence $c(\Delta)\leqslant\frac{2\pi}{15}$.*

Proof. Observe that $d(v_a)\geqslant 4$; moreover $d(v_i)\neq 3$ for $i\in\{b,c,d,e\}$. It follows that if at least three of the vertices v_b, v_c, v_d and v_e have degree >2 then $c(\Delta)\leqslant c(4,4,4,4)=0$.

Suppose that exactly two of the vertices v_b, v_c, v_d and v_e have degree 2. Observe that a splitting cannot have degree 3, therefore we can assume there is no splitting in Δ otherwise $c(\Delta)\leqslant c(4,4,4,4)=0$. Moreover since $|ab|<|de|$ and $|ea|<|bc|$, we can assume $d(v_d)\geqslant 4$ and $d(v_c)\geqslant 4$ otherwise de splits or bc splits, respectively. It follows that we must have $d(v_b)=d(v_e)=2$. Since we are assuming there is no splitting in Δ it follows that v_a has sublabel $\mu a\mu$ which implies $d(v_a)\geqslant 5$. Now the segment cd has maximal length so it follows from Lemma 3.1(vi) that the adjacent region along this segment must be an s-region and so that v_c and v_d have sublabels ccs and sdd, respectively. Using the star graph one can easily verify the following.

If $d(v_c)=5$ one the following holds:

$$i)\quad \begin{aligned} &2\beta+\alpha=0;\\ &2\beta-\alpha=0;\\ &2\beta+\alpha-1=0;\\ &2\beta-\alpha-1=0.\end{aligned}$$

If $d(v_d)=5$ one the following holds:

$$ii)\quad \begin{aligned} &2\alpha+\beta=0;\\ &2\alpha-\beta=0;\\ &2\alpha+\beta+1=0;\\ &2\alpha-\beta+1=0.\end{aligned}$$

It follows that if one of the vertices v_c and v_d has degree 5 then the other one has degree $\geqslant 6$ hence $c(\Delta)\leqslant c(5,5,6)=\frac{2\pi}{15}$ and Δ is given by Fig. 4(i).

Now suppose that exactly three of the vertices v_b, v_c, v_d and v_e have degree 2. If $d(v_b)>2$ or $d(v_e)>2$ it is easy to see that the segments bc, cd and de split, hence $c(\Delta)\leqslant c(4,4,4,4,4)<0$. So we can assume $l(v_b)=bd$ and $l(v_e)=ec$. If $d(v_c)>2$ and $d(v_d)=2$ then $l(v_d)=db$ and $d(v_c)\geqslant 4$. Since $|ab|<|bc|$ and $|cd|=|de|$ it follows that de splits and the splitting v has sublabel $\{a,\lambda\}\mu\lambda$ which implies $d(v)\geqslant 6$. Since there can be no other splitting, it follows that v_a and v_c have sublabels $\mu a\mu$ and ccc respectively, which imply $d(v_a)\geqslant 5$ and $d(v_c)\geqslant 6$. Thus $c(\Delta)\leqslant c(5,6,6)=\frac{\pi}{15}$ and Δ is given by Fig. 4(ii). If $d(v_c)=2$ and $d(v_d)>2$ then $l(v_c)=ce$ and $d(v_d)\geqslant 4$. Since $|ea|<|bc|$ and $|cd|=|bc|$ it follows that bc splits and the splitting v has sublabel $\lambda\mu\{a,\lambda\}$ which implies $d(v)\geqslant 6$. Since there can be no other splitting, it follows that v_a and v_d have sublabels $\mu a\mu$ and ddd respectively, which imply $d(v_a)\geqslant 5$ and $d(v_d)\geqslant 6$. Thus $c(\Delta)\leqslant c(5,6,6)=\frac{\pi}{15}$ and Δ is given by Fig. 4(iii).

Finally suppose $d(v_b)=d(v_c)=d(v_d)=d(v_e)=2$, hence $l(v_b)=l(v_d)=bd$ and $l(v_c)=l(v_e)=ce$. Since $|ea|<|bc|$, $|ab|<|de|$ and be is not a segment label it follows that the segments bc, de and cd split and so $c(\Delta)\leqslant c(4,4,4,4)=0$. □

Distribution of curvature

Now we describe the distribution of curvature. When $j<\frac{n}{3}$ then by Lemmas 3.2 and 3.3 there is no positively curved interior region, thus there is no distribution of curvature and $c^*(\Delta)=c(\Delta)$ for each region Δ.

Lemma 3.5. *If $j<\frac{n}{3}$ and Δ is a boundary region of $\mathscr{D}$ then $c^*(\Delta)=c(\Delta)<\frac{4\pi}{k_0}$.*

Proof. Let Δ be a boundary s-region. Since t has order n in the extension E, it follows that each consequence of the relators must have exponent sum of t congruent to 0 modulo n; this implies

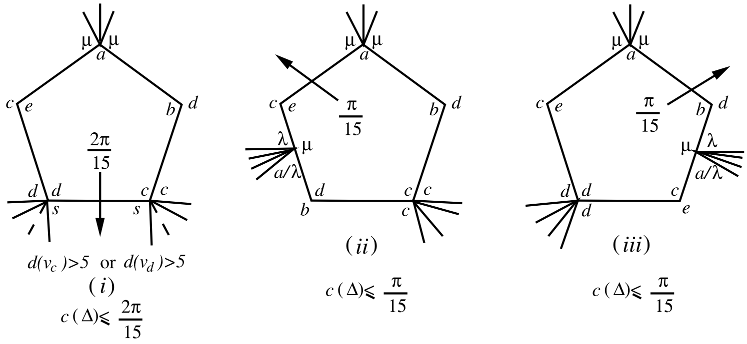

Fig. 4. positively curved interior m-region and distribution of curvature for $j - i > \frac{n}{2}$.

that the distinguished vertex coincides with exactly one vertex of Δ. Since $d(\Delta) \geqslant 4$ it follows that $c^*(\Delta) = c(\Delta) \leqslant c(3, 3, 3, k_0) = \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$.

Now let Δ be a boundary m-region. Suppose that the distinguished vertex v_0 coincides with m vertices of Δ and so $k_0 \geqslant 2m$. The exponent sum of t in each closed path label must be congruent to 0 modulo n. This implies that v_0 cannot coincide with more than one vertex for each given segment, so $m \leqslant 5$; and since $|ab| + |bc| = j < n$ and they have the same orientation it follows that $m \leqslant 4$. Note also that if $v_a = v_0$ then $m = 1$.

If $m = 4$ then $c^*(\Delta) = c(\Delta) \leqslant c(k_0, k_0, k_0, k_0) = -2\pi + \frac{4\pi}{k_0} + \frac{4\pi}{k_0} \leqslant -2\pi + \frac{\pi}{2} + \frac{4\pi}{k_0} < \frac{4\pi}{k_0}$; if $m = 3$ then $c^*(\Delta) = c(\Delta) \leqslant c(k_0, k_0, k_0) = -\pi + \frac{2\pi}{k_0} + \frac{4\pi}{k_0} \leqslant -\pi + \frac{\pi}{3} + \frac{4\pi}{k_0} < \frac{4\pi}{k_0}$; if $m = 2$ then $v_a \neq v_0$ and it follows that $c^*(\Delta) = c(\Delta) \leqslant c(3, k_0, k_0) = -\frac{\pi}{3} + \frac{4\pi}{k_0} < \frac{4\pi}{k_0}$; and if $m = 1$ then since v_a, v_b, v_d, if interior, each have degree > 2 and since $d(v_e) = 2$ forces ea to split, it follows that $c^*(\Delta) = c(\Delta) \leqslant c(3, 3, 3, k_0) = \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$. $\square$

Let us consider the case (3) with the additional assumption that each s-region in $\mathscr{D}$ has degree $\geqslant 4$. Then by Lemma 3.2 and Lemma 3.4 there is no positively curved interior s-region and the positively curved interior m-regions are given by Fig. 4 (i)–(iii). Let Δ be a positively curved interior m-region; we distribute the curvature according to Fig. 4. Observe that if Δ is an m-region receiving curvature then either the segments bc and cd both split or the segments cd and de both do, in particular $c(\Delta) \leqslant 0$.

Lemma 3.6. *If $j - i > \frac{n}{2}$ and Δ is an interior region of $\mathscr{D}$ then $c^*(\Delta) \leqslant 0$.*

Proof. It is clear from the distribution of curvature that we need consider only those regions which receive positive curvature according to Fig. 4. Let Δ be such a region.

If Δ is an s-region receiving curvature from k segments then each of these segments has one end vertex of degree $\geqslant 5$ and one end vertex of degree $\geqslant 6$; moreover the incoming curvature through each segment is $\frac{2\pi}{15}$. Since $d(\Delta) \geqslant 4$ and there are no interior vertices of degree 3 it follows that $c^*(\Delta) = c(\Delta) + \frac{2\pi}{15}k \leqslant -\pi(d(\Delta) - 2) + \frac{\pi}{2}(d(\Delta) - k - 1) + \frac{k+1}{2}\frac{2\pi}{5} + \frac{k+1}{2}\frac{\pi}{3} + k\frac{2\pi}{15} = -\frac{\pi}{2}d(\Delta) + \frac{28\pi}{15} \leqslant -2\pi + \frac{28\pi}{15} < 0$.

Let Δ be an m-region receiving curvature as in Fig. 4(ii) and, in addition, possibly Fig. 4(iii), then the segments bc and cd both split and the splittings have at least degree 5 and 6, respectively. Since we also have $d(v_a) \geqslant 4$ and $d(v_e) = d(v_d) = 2$ forces a splitting, it follows that $c(\Delta) \leqslant c(4, 4, 5, 6) = -\frac{4\pi}{15}$. The maximum total amount of curvature that Δ can receive is $\frac{\pi}{15} + \frac{\pi}{15} = \frac{2\pi}{15}$, hence $c^*(\Delta) < 0$.

It remains to check those m-regions receiving curvature as in Fig. 4 (iii) only. In this case the segments cd and de both split in degree $\geqslant 6$ and 5, respectively; moreover $d(v_a) \geqslant 4$ and if $d(v_c) < 4$ then the segment bc splits. Thus $c^*(\Delta) \leqslant c(\Delta) + \frac{\pi}{15} \leqslant c(4, 4, 5, 6) + \frac{\pi}{15} < 0$. $\square$

Lemma 3.7. *If $j - i > \frac{n}{2}$ and Δ is a boundary region of $\mathcal{D}$ then $c^*(\Delta) < \frac{4\pi}{k_0}$.*

Proof. Let Δ be a boundary s-region. Then Δ receives at most the positive curvature $\frac{2\pi}{15}$ through each interior segment and we can assume that the distinguished vertex coincides with exactly one vertex of Δ, so that there are $d(\Delta) - 2$ interior segments. It follows that $c^*(\Delta) \leqslant c(\Delta) + (d(\Delta) - 2)\frac{2\pi}{15} = (d(\Delta) - 2)(-\pi + \frac{2\pi}{15}) + (d(\Delta) - 1)\frac{\pi}{2} + \frac{2\pi}{k_0}$; since $d(\Delta) \geqslant 4$ it follows that $c^*(\Delta) < \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$.

Now let Δ be a boundary m-region. The maximum total amount of curvature that Δ can receive is $\frac{\pi}{15} + \frac{\pi}{15} = \frac{2\pi}{15}$. Suppose that the distinguished vertex v_0 coincides with m vertices of Δ. Observe that $k_0 \geqslant 2m$. As before $m \leqslant 5$. Since $|ea| + |ab| = n - j + i$ and they have the same orientation it follows that $m \leqslant 4$.

If $m = 4$ then $c^*(\Delta) \leqslant c(k_0, k_0, k_0, k_0) + \frac{2\pi}{15} = -2\pi + \frac{8\pi}{k_0} + \frac{2\pi}{15} \leqslant -2\pi + \pi + \frac{2\pi}{15} < 0 < \frac{4\pi}{k_0}$; if $m = 3$ then $c^*(\Delta) \leqslant c(\Delta) + \frac{2\pi}{15} \leqslant c(k_0, k_0, k_0) + \frac{2\pi}{15} = -\pi + \frac{2\pi}{k_0} + \frac{4\pi}{k_0} + \frac{2\pi}{15} \leqslant -\pi + \frac{\pi}{3} + \frac{4\pi}{k_0} + \frac{2\pi}{15} < \frac{4\pi}{k_0}$; if $m = 2$ then Δ must have two interior vertices of degree $\geqslant 4$ therefore $c^*(\Delta) \leqslant c(\Delta) + \frac{2\pi}{15} \leqslant c(4, 4, k_0, k_0) + \frac{2\pi}{15} = -\pi + \frac{4\pi}{k_0} + \frac{2\pi}{15} < \frac{4\pi}{k_0}$. Suppose finally that $m = 1$ and note that $d(\Delta) \geqslant 3$. If $d(\Delta) \geqslant 4$ then $c^*(\Delta) \leqslant c(\Delta) + \frac{2\pi}{15} \leqslant c(4, 4, 4, k_0) + \frac{2\pi}{15} = -2\pi + \frac{3\pi}{2} + \frac{2\pi}{k_0} + \frac{2\pi}{15} < \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$; and if $d(\Delta) = 3$ then Δ has at least two interior vertices of degree $\geqslant 5$ therefore $c^*(\Delta) \leqslant c(\Delta) + \frac{2\pi}{15} \leqslant c(5, 5, k_0) + \frac{2\pi}{15} = -\pi + \frac{4\pi}{5} + \frac{2\pi}{k_0} + \frac{2\pi}{15} < \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$. □

Proposition 3.8. *Let $G = G_n(x_0[x_i^\alpha, x_j^\beta])$ and let E be the split extension of G given in Section 2.*

(i) *If $j < \frac{n}{3}$ then x has infinite order in E.*
(ii) *If $j - i > \frac{n}{2}$ and x has finite order in E then there is a modified van Kampen diagram $\mathcal{D}$ (constructed as in the above) containing an s-region of degree < 4.*

Proof. (i) follows from Lemmas 3.1, 3.2, 3.3 and 3.5 as explained in Section 2. When $j - i > \frac{n}{2}$ it follows from Lemmas 3.1, 3.2, 3.4, 3.5 and 3.7 that under the additional assumption **A4** the total curvature T of the modified van Kampen diagram $\mathcal{D}$ is less than 4π, a contradiction. Thus $|x| < \infty$ forces $\mathcal{D}$ to contain an s-region of degree < 4. □

Proof of Corollary 1.3. If $j < \frac{n}{3}$ the corollary is obvious. The assumption $j - i > \frac{2n}{3}$ implies that we are in case (3) and that any modified van Kampen diagram $\mathcal{D}$ has maximal m-segment length strictly less than $\frac{n}{3}$. Since $\mathcal{D}$ is reduced it follows that s-regions cannot be adjacent to each other therefore each s-region has degree $\geqslant 4$ and Proposition 3.8 applies. □

In order to complete the proof of Theorem 1.1 in Section 4 we will need to consider the cases $j \in \{\frac{n}{3}, \frac{n}{3} + 1, \frac{2n}{3}, \frac{2n}{3} + 1\}$ and in doing this we no longer make the assumption that in case $j - i > \frac{n}{2}$ each s-region in $\mathcal{D}$ has degree $\geqslant 4$.

Proposition 3.9. *Let $n = 3n_1$, $n \geqslant 12$. Then the groups $G_n(x_0[x_1^\alpha, x_{n_1}^\beta])$ and $G_n(x_0[x_1^\alpha, x_{2n_1}^\beta])$ are infinite.*

Proof. Let $G \cong G_n(x_0[x_1^\alpha, x_{n_1}^\beta])$ and consider the extension $E(n; \alpha, \beta; 1, n_1)$ of G. Then

$$\begin{aligned} E(n; \alpha, \beta; 1, n_1) &= \langle x, t \mid t^n, xt^{-1}x^{-\alpha}t^{1-n_1}x^{-\beta}t^{n_1-1}x^\alpha t^{1-n_1}x^\beta t^{n_1} \rangle \\ &\twoheadrightarrow \langle x, t \mid t^{n_1}, t^{-1}x^{-\alpha}tx^{-\beta}t^{-1}x^\alpha tx^{\beta+1} \rangle = E(n_1; \alpha, \beta; 1, 0) \end{aligned}$$

where the second relator in the last presentation is cyclically reduced since $\beta \neq -1$. This is an extension of $G_{n_1}(x_0[x_1^\alpha, x_0^\beta])$ which is infinite since $n_1 \geqslant 4$ and each relator involves only two generators (see [12] and Section 2). The other case is similar. □

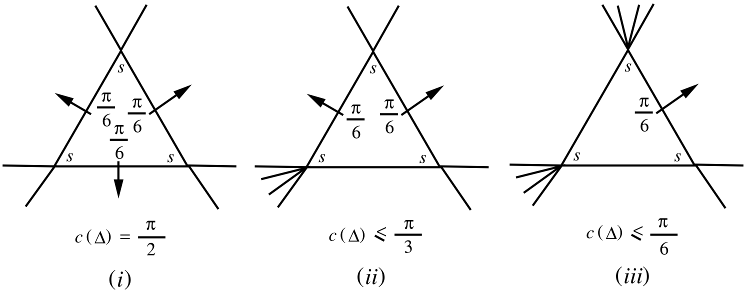

Fig. 5. Distribution for s-regions of degree 3 and at least one vertex of degree 4.

Proposition 3.10. *Let $n = 3n_1$, $n \geqslant 12$. Then the groups $G_n(x_0[x_2^\alpha, x_{2n_1+2}^\beta])$ and $G_n(x_0[x_1^\alpha, x_{2n_1+1}^\beta])$ are infinite.*

Proof. As before we will prove that x has infinite order in the corresponding extension E. Thus assume, by way of contradiction, that $|x| < \infty$ and let $\mathscr{D}$ be the corresponding tessellation of the 2-sphere. In both cases the given presentation is in case (3), that is $j - i > \frac{n}{2}$; moreover the maximal segment length in an m-region is $n - j + i = \frac{n}{3}$. The positively curved interior m-regions are given by Lemma 3.4 and Fig. 4.

Now let Δ be an interior s-region of degree 3. Since Δ is adjacent to m-regions only and since the maximal segment length in an m-region is exactly $\frac{n}{3}$, it follows that the three segments of Δ are all segments of maximal length in the adjacent m-regions. Thus, up to inversion, the possible sublabels of vertices of Δ are given by $\{b, c, d, \bar{c}, \bar{d}, \bar{e}\}s\{c, d, e, \bar{b}, \bar{c}, \bar{d}\}$. An analysis of the star graph shows that if there is a vertex of degree 4 one of the following holds

$$i)\quad \alpha \pm \beta \pm 1 = 0;$$

and if there is a vertex of degree 5 one of the following holds

$$ii)\quad \begin{aligned} &\alpha \pm 2\beta = 0;\\ &2\alpha \pm \beta = 0;\\ &\alpha \pm 2\beta \pm 1 = 0;\\ &2\alpha \pm \beta \pm 1 = 0. \end{aligned}$$

Since each of the identities in i) is mutually exclusive with any of the identities in ii) it follows that the occurrence of a vertex of degree 4 excludes the presence of vertices of degree 5 in an s-region of degree 3 and conversely. Similarly, a check of the possible labels and their consequences for α and β shows that if Δ has a vertex of degree 5 then there is no vertex of degree 6 in Δ.

If there is a vertex of degree 4 the distribution of curvature is given by Fig. 5 and if there is a vertex of degree 5 by Fig. 6.

Consider the m-regions of Fig. 4 (i)–(iii). If $|\alpha| > 2$ and $|\beta| > 2$ then $d(v_a) \geqslant 6$ and Δ can be positively curved only for Fig. 4(i). In this case v_c and v_d have sublabels ccs and sdd respectively; we have already proved that if v_c has degree 5 then v_d has degree $\geqslant 6$ and conversely if v_d has degree 5 then v_c has degree $\geqslant 6$. Since $c(6, 6, 6) = 0$ we can assume that Δ has a vertex of degree 5. An analysis of the star graph shows that $d(v_c) = 5$ implies $d(v_d) \geqslant 7$ and that $d(v_d) = 5$ implies $d(v_c) \geqslant 7$. Since $c(5, 7, 7) < 0$ and $c(5, 6, 8) < 0$ it follows that Δ can be positively curved only if $d(v_a) = 6$ and $\{d(v_c), d(v_d)\} = \{5, 7\}$. A check shows that $(d(v_c), d(v_d)) = (5, 7)$ can occur only for $(\alpha, \beta) = (-5, 3)$ and that $(d(v_c), d(v_d)) = (7, 5)$ can occur only for $(\alpha, \beta) = (-3, 5)$. Observe that in both cases Δ cannot receive positive curvature from the s-region adjacent to the segment cd and in this case the curvature is distributed according to Fig. 7 (i) and (ii).

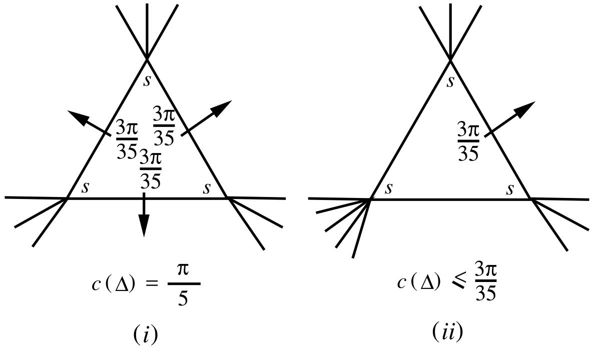

Fig. 6. Distribution for s-regions of degree 3 and at least one vertex of degree 5.

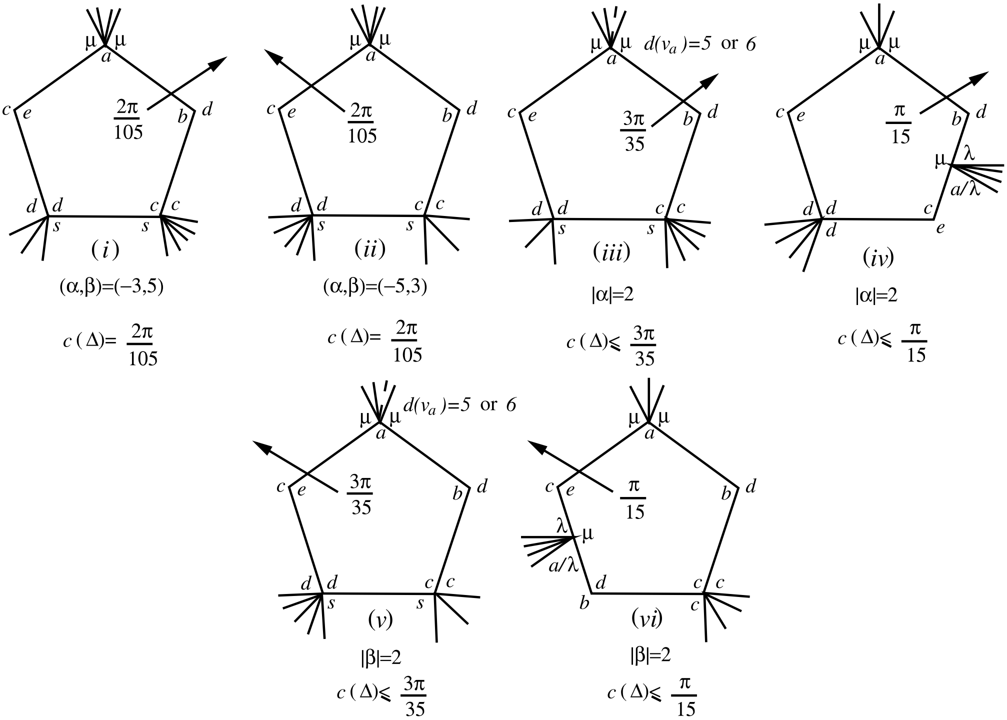

Fig. 7. Distribution of curvature for interior m-regions.

Now suppose $|\alpha| = 2$. In Fig. 4(i) v_c has sublabel ccs and it follows that $d(v_c) \geqslant 7$. In Fig. 4(ii), v_c has sublabel ccc which implies $d(v_c) \geqslant 8$ hence $c(\Delta) \leqslant c(5,6,8) < 0$.

Similarly for $|\beta| = 2$, in Fig. 4 (i) we have $d(v_d) \geqslant 7$ and in Fig. 4(iii) we have $d(v_d) \geqslant 8$ and $c(\Delta) < 0$.

In these cases the curvature is distributed according to Fig. 7 (iii)–(iv) and Fig. 7 (v)–(vi), respectively.

To complete the proof as before it will be sufficient to prove the following two statements.

(i) If Δ is interior then $c^*(\Delta) \leqslant 0$;
(ii) if Δ is a boundary region then $c^*(\Delta) < \frac{4\pi}{k_0}$.

Let Δ be interior. If Δ is an s-region then it is clear that $c^*(\Delta) \leqslant 0$ so it can be assumed that Δ is an m-region. Suppose first of all that Δ receives positive curvature from an m-region (see Fig. 7) and note that in this case Δ can in addition receive positive curvature of at most $\frac{\pi}{6}$ from at most one s-region (since curvature is transferred across maximal length segments).

Let $(\alpha, \beta) = (-3, 5)$. Then as shown in Fig. 7(i) the segment de splits in degree 6, $d(v_c) = 7$ and $d(v_a) \geqslant 4$. If $d(v_e) = 2$ and there is no other splitting in de, then the splitting in de has sublabel $\mu a \mu \lambda$ contradicting the fact that it has degree 6. It follows that there is another vertex of degree $\geqslant 4$ and so $c^*(\Delta) \leqslant c(4, 4, 6, 7) + \frac{2\pi}{105} + \frac{\pi}{6} < 0$.

Let $(\alpha, \beta) = (-5, 3)$. The segment bc splits in degree 6, $d(v_d) = 7$ and $d(v_a) \geqslant 4$. If $d(v_b) = 2$ and there is no other splitting in bc, then the splitting in bc has sublabel $\lambda \mu a \mu$ contradicting the fact that it has degree 6. It follows that there is another vertex of degree $\geqslant 4$ and so $c^*(\Delta) \leqslant c(4, 4, 6, 7) + \frac{2\pi}{105} + \frac{\pi}{6} < 0$.

Let $|\alpha| = 2$. The segment de splits in degree 5 or 6, there is another vertex of degree $\geqslant 6$ and $d(v_a) \geqslant 4$. If $d(v_e) = 2$ and there is no other splitting in de, then the splitting in de has sublabel $\mu a \mu \lambda$ contradicting the fact that it has degree 5 or 6. It follows that there is another vertex of degree $\geqslant 4$ and so $c^*(\Delta) \leqslant c(4, 4, 5, 6) + \frac{3\pi}{35} + \frac{\pi}{6} = -\frac{4\pi}{15} + \frac{3\pi}{35} + \frac{\pi}{6} < 0$.

Let $|\beta| = 2$. The segment bc splits in degree 5 or 6, there is another vertex of degree $\geqslant 6$ and $d(v_a) \geqslant 4$. If $d(v_b) = 2$ and there is no other splitting in bc, then the splitting in bc has sublabel $\lambda \mu a \mu$ contradicting the fact that it has degree 5 or 6. It follows that there is another vertex of degree $\geqslant 4$ and so $c^*(\Delta) \leqslant c(4, 4, 5, 6) + \frac{3\pi}{35} + \frac{\pi}{6} = -\frac{4\pi}{15} + \frac{3\pi}{35} + \frac{\pi}{6} < 0$.

Now let Δ receive curvature from s-regions only. The fact that curvature is transferred through the segments bc, cd and de only, implies that Δ receives from at most three regions.

Let Δ receive positive curvature from three s-regions. Then $d(v_i) \geqslant 4$ for $i = a, b, c, d, e$, therefore $c^*(\Delta) \leqslant c(4, 4, 4, 4, 4) + \frac{\pi}{2} = 0$.

Let Δ receive positive curvature from exactly two s-regions. The three possibilities are through segments bc and cd, bc and de or cd and de.

If through bc and cd then $d(v_b) \geqslant 4$, $d(v_c) \geqslant 4$ and $d(v_d) \geqslant 4$. The maximum total amount of curvature that Δ can receive is $\frac{\pi}{3}$. If $d(v_e) > 2$ or there is a splitting in Δ then $c^*(\Delta) \leqslant c(4, 4, 4, 4, 4) + \frac{\pi}{3} < 0$; so assume that $l(v_e) = ec$ and the segments de and ea do not split. This implies that v_d and v_a have sublabels sdd and μa respectively, hence $d(v_d) \geqslant 5$ and $d(v_a) \geqslant 5$. Furthermore the vertex v_c has sublabel $\bar{s}cs$, which implies $d(v_c) \geqslant 5$. If $\frac{3\pi}{35}$ is transferred through each segment then we also have $d(v_b) \geqslant 5$ and $c^*(\Delta) \leqslant c(5, 5, 5, 5) + \frac{6\pi}{35} < 0$. If $\frac{\pi}{6}$ is transferred through each segment then $d(v_c) \geqslant 6$ and $d(v_d) \geqslant 6$, therefore $c^*(\Delta) \leqslant c(4, 5, 6, 6) + \frac{\pi}{3} < 0$.

If through bc and de then $d(v_b) \geqslant 4$, $d(v_c) \geqslant 4$, $d(v_d) \geqslant 4$ and $d(v_e) \geqslant 4$, hence $c^*(\Delta) \leqslant c(4, 4, 4, 4, 4) + \frac{\pi}{3} < 0$.

If through cd and de then $d(v_c) \geqslant 4$, $d(v_d) \geqslant 4$ and $d(v_e) \geqslant 4$. The maximum total amount of curvature that Δ can receive is $\frac{\pi}{3}$. If $d(v_b) > 2$ or there is a splitting in Δ then $c^*(\Delta) \leqslant c(4, 4, 4, 4, 4) + \frac{\pi}{3} < 0$; so assume that $l(v_b) = bd$ and the segments ab and bc do not split. This implies that v_c and v_a have sublabels ccs and $a\mu$ respectively, hence $d(v_c) \geqslant 5$ and $d(v_a) \geqslant 5$. Furthermore the vertex v_d has sublabel $sd\bar{s}$, which implies $d(v_d) \geqslant 5$. If $\frac{3\pi}{35}$ is transferred through each segment then we also have $d(v_e) \geqslant 5$ and $c^*(\Delta) \leqslant c(5, 5, 5, 5) + \frac{6\pi}{35} < 0$. If $\frac{\pi}{6}$ is transferred through each segment then $d(v_c) \geqslant 6$ and $d(v_d) \geqslant 6$, therefore $c^*(\Delta) \leqslant c(4, 5, 6, 6) + \frac{\pi}{3} < 0$.

Now assume that Δ receives positive curvature from exactly one s-region; either $\frac{\pi}{6}$ or $\frac{3\pi}{35}$ through bc, cd or de.

Suppose Δ receives $\frac{\pi}{6}$ through bc. If $d(v_e) = d(v_d) = 2$ then the segment de splits, therefore there is another vertex of degree $\geqslant 4$. We can assume that $d(v_b) = d(v_c) = 4$, for if not v_b or v_c has degree $\geqslant 6$ and $c^*(\Delta) \leqslant c(4, 4, 4, 6) + \frac{\pi}{6} = 0$. But $d(v_c) = 4$ implies that v_c has sublabel $\bar{s}c\bar{a}$; since $|\bar{b}\bar{a}| < |cd|$ it follows that cd splits and $c^*(\Delta) \leqslant c(4, 4, 4, 4, 4) + \frac{\pi}{6} < 0$.

Suppose Δ receives $\frac{3\pi}{35}$ through bc. If $d(v_e) = d(v_b) = 2$ then the segment de splits, therefore there is another vertex of degree $\geqslant 4$; moreover we have $d(v_b) \geqslant 5$ and $d(v_c) \geqslant 5$. It follows that $c^*(\Delta) \leqslant c(4, 4, 5, 5) + \frac{3\pi}{35} < 0$.

Suppose Δ receives $\frac{\pi}{6}$ through cd. If $d(v_c)=4$ then bc splits and if $d(v_d)=4$ then de splits. We can therefore assume $d(v_c)=d(v_d)=4$ for if not then $c^*(\Delta)\leqslant c(4,4,4,6)+\frac{\pi}{6}=0$. It follows that bc and de both split and $c^*(\Delta)\leqslant c(4,4,4,4,4)+\frac{\pi}{6}<0$.

Suppose Δ receives $\frac{3\pi}{35}$ through cd. Then $d(v_c)=d(v_d)=5$, but this implies that either bc or de splits, hence $c^*(\Delta)\leqslant c(4,4,5,5)+\frac{3\pi}{35}<0$.

Now suppose Δ receives $\frac{\pi}{6}$ through de. If $d(v_b)=d(v_c)=2$ then the segment bc splits, therefore there is another vertex of degree $\geqslant 4$. Now we can assume that $d(v_d)=d(v_e)=4$, for if not v_d or v_e has degree $\geqslant 6$ and $c^*(\Delta)\leqslant c(4,4,4,6)+\frac{\pi}{6}=0$. But $d(v_d)=4$ implies that v_d has sublabel $\bar{a}d\bar{s}$; since $|\bar{a}\bar{e}|<|cd|$ it follows that cd splits and $c^*(\Delta)\leqslant c(4,4,4,4,4)+\frac{\pi}{6}<0$.

Finally suppose Δ receives $\frac{3\pi}{35}$ through de. If $d(v_b)=d(v_c)=2$ then the segment bc splits, therefore there is another vertex of degree $\geqslant 4$; moreover we have $d(v_d)\geqslant 5$ and $d(v_e)\geqslant 5$. It follows that $c^*(\Delta)\leqslant c(4,4,5,5)+\frac{3\pi}{35}<0$.

So let Δ be a boundary region. If Δ is an s-region then the distinguished vertex cannot coincide with more than one vertex in Δ. Moreover Δ does not receive any positive curvature from adjacent regions. It follows that $c^*(\Delta)=c(\Delta)\leqslant c(4,4,k_0)=\frac{2\pi}{k_0}<\frac{4\pi}{k_0}$.

Now let Δ be a boundary m-region. Suppose that the distinguished vertex coincides with m vertices of Δ so that $k_0\geqslant 2m$. As in the above $m\leqslant 5$. If $m\geqslant 3$ then $c^*(\Delta)\leqslant c(\Delta)+\frac{\pi}{2}\leqslant c(k_0,k_0,k_0)+\frac{\pi}{2}=-\pi+\frac{2\pi}{k_0}+\frac{4\pi}{k_0}+\frac{\pi}{2}\leqslant -\pi+\frac{\pi}{3}+\frac{4\pi}{k_0}+\frac{\pi}{2}<\frac{4\pi}{k_0}$.

If $m=2$ then Δ cannot receive $\frac{\pi}{2}$ therefore $c^*(\Delta)<c(\Delta)+\frac{\pi}{2}\leqslant c(4,k_0,k_0)+\frac{\pi}{2}=\frac{4\pi}{k_0}$.

If $m=1$ and there are at least three interior vertices of degree $\geqslant 4$ then $c^*(\Delta)\leqslant c(\Delta)+\frac{\pi}{2}\leqslant c(4,4,4,k_0)+\frac{\pi}{2}=-2\pi+\frac{3\pi}{2}+\frac{2\pi}{k_0}+\frac{\pi}{2}=\frac{2\pi}{k_0}<\frac{4\pi}{k_0}$. If there are only two interior vertices then there is an interior vertex of degree $\geqslant 6$ and the maximum amount of curvature that Δ can receive is $\frac{\pi}{6}$; it follows that $c^*(\Delta)\leqslant c(\Delta)+\frac{\pi}{6}\leqslant c(4,6,k_0)+\frac{\pi}{6}=-\pi+\frac{5\pi}{6}+\frac{2\pi}{k_0}+\frac{\pi}{6}=\frac{2\pi}{k_0}<\frac{4\pi}{k_0}$. □

4. Proof of Theorems 1.1 and 1.2

We will need the following.

Lemma 4.1. *Let $n\in\mathbb{N}$ be odd, $n\geqslant 11$ and $j\in\mathbb{Z}_n$, $j\neq 0$ or 1 modulo n, $j\notin\{\frac{n}{3},\frac{n}{3}+1,\frac{2n}{3},\frac{2n}{3}+1\}$. Then there exists an automorphism of $\mathbb{Z}_n$ sending $(1,j)$ to (i',j') such that one of the following is satisfied*:

(i) $i', j'<\frac{n}{3}$;
(ii) $i'<\frac{n}{3}$, $n-j'+i'<\frac{n}{3}$.

Proof. First assume n is coprime to 3 so that $1\mapsto 2$ and $1\mapsto 3$ define automorphisms of $\mathbb{Z}_n$. If $j<\frac{n}{3}$ there is nothing to prove; if $\frac{n}{3}<j<\frac{4n}{9}$ apply $1\mapsto 3$ to obtain $i'=3<\frac{n}{3}$ and $j'=3j-n<\frac{n}{3}$; if $\frac{4n}{9}<j<\frac{n}{2}$ apply $1\mapsto 2$ to obtain $i'=2<\frac{n}{3}$, $j'=2j$ so that $n-j'+i'<n-\frac{8n}{9}+2<\frac{n}{3}$; if $\frac{n}{2}<j<\frac{2n}{3}$ apply $1\mapsto 2$ to obtain $i'=2<\frac{n}{3}$ and $j'=2j-n<\frac{n}{3}$ since $j<\frac{2n}{3}$; if $\frac{2n}{3}<j<\frac{7n}{9}$ apply $1\mapsto 3$ to obtain $i'=3<\frac{n}{3}$ and $j'=3j-2n<\frac{n}{3}$ since $j<\frac{7n}{9}$ and finally if $j>\frac{7n}{9}$ then $n-j+1<n-\frac{7n}{9}+1<\frac{n}{3}$.

Now suppose that 3 divides n. If $j<\frac{n}{3}$ there is nothing to prove; if $\frac{n}{3}+1<j<\frac{n}{2}$ apply $1\mapsto 2$ to obtain $i'=2<\frac{n}{3}$ and $n-j'+i'=n-2j+2<n-\frac{2n}{3}-2+2=\frac{n}{3}$; if $\frac{n}{2}<j<\frac{2n}{3}$ apply $1\mapsto 2$ to obtain $i'=2<\frac{n}{3}$ and $j'=2j-n<\frac{n}{3}$ since $j<\frac{2n}{3}$ and finally if $j>\frac{2n}{3}+1$ then $n-j+1<n-\frac{2n}{3}-1+1=\frac{n}{3}$. □

Let $G=G_n(x_0[x_i^\alpha,x_j^\beta])$. Lemma 4.1 assures us that when $i=1$ there exists a sequence of elementary moves mapping the given presentation to another which satisfies the hypotheses of Corollary 1.3. In view of this we are now able to prove the main theorem.

Proof of Theorem 1.1. Put $h_i=\mathrm{hcf}(n,i)$ and $h_j=\mathrm{hcf}(n,j)$. The assumption $h_i=1$ or $h_j=1$ implies that there is a sequence of elementary moves yielding $G=G_n(x_0[x_i^\alpha,x_j^\beta])\cong G_n(x_0[x_1^{\alpha'},x_{j'}^{\beta'}])$ where

$(\alpha', \beta') \in \{(\alpha, \beta), (-\beta, -\alpha)\}$ and $n \neq 2(j'-1)$. So we can assume without loss of generality that $G = G_n(x_0[x_1^\alpha, x_j^\beta])$.

First suppose that $k = 0$ so that $n = m$ is odd. If $j \in \{\frac{n}{3}, \frac{2n}{3}, \frac{2n}{3} + 1\}$ then G is infinite by Propositions 3.9 and 3.10. If $j = \frac{n}{3} + 1$ then the automorphism $1 \mapsto 2$ applies since n is odd and $G = G_n(x_0[x_2^{\alpha'}, x_{2n_1+2}^{\beta'}])$ where $n_1 = \frac{n}{3}$ and so G is infinite by Proposition 3.10. Otherwise apply Lemma 4.1 to map $G_n(x_0[x_1^\alpha, x_j^\beta])$ to another such presentation satisfying (i) or (ii) of Lemma 4.1. The result follows from Corollary 1.3.

Now suppose that $k > 0$. If $m \nmid j - i$ then

$$E = E(n; \alpha, \beta; 1, j) = \langle x, t \mid t^n, xt^{-1}x^{-\alpha}t^{1-j}x^{-\beta}t^{j-1}x^\alpha t^{1-j}x^\beta t^j \rangle$$
$$\twoheadrightarrow \langle x, t \mid t^m, xt^{-1}x^{-\alpha}t^{1-j'}x^{-\beta}t^{j'-1}x^\alpha t^{1-j'}x^\beta t^{j'} \rangle,$$

where $j' - 1 \neq 0 \pmod m$ and $j' = j \pmod m$. If $j' \neq 0 \pmod m$ then we are back in case $k = 0$; otherwise we obtain $E(m; \alpha, \beta; 1, 0)$ which is infinite [12].

Assume therefore that $m \mid j - 1$. Then $j - 1 \in \{m, 2m, \ldots, (2^k - 1)m\} \setminus \{2^{k-1}m\}$ so in particular $k \geqslant 2$. If $j - 1 \in \{m, \ldots, \lfloor \frac{2^k}{3} \rfloor m\}$ then $j < \frac{n}{3}$ or if $j - 1 \in \{\lceil \frac{2^{k+1}}{3} \rceil m, \ldots, (2^k - 1)m\}$ then $j > \frac{2n}{3}$ and the result follows from Corollary 1.3. This leaves $j - 1 \in \{\lceil \frac{2^k}{3} \rceil m, \ldots, (2^{k-1} - 1)m\} \cup \{(2^{k-1} + 1)m, \ldots, \lfloor \frac{2^{k+1}}{3} \rfloor m\}$. Now map E onto $E(2^{k-1}m; \alpha, \beta; 1, j')$ noting that $2^{k-1}m$ does not divide $j - 1$. If $2^{k-1}m$ divides j' then E is infinite as before so suppose otherwise. All that remains is to observe that if $j - 1$ is in the first of the two sets above then $j' - 1 > \frac{2^k m}{3} = \frac{2}{3}(2^{k-1}m)$; or if in the second then $j' < \frac{2^{k-1}m}{3}$ and the result follows from Corollary 1.3. $\square$

Proof of Theorem 1.2. Suppose $h_i > 3$. Since G is irreducible, that is $\mathrm{hcf}(n, i, j) = 1$, it follows that $\mathrm{hcf}(j, h_i) = 1$ and $h_i \nmid j - i$. Then

$$E(n; \alpha, \beta; i, j) = \langle x, t \mid t^n, xt^{-i}x^{-\alpha}t^{i-j}x^{-\beta}t^{j-i}x^\alpha t^{i-j}x^\beta t^j \rangle$$
$$\twoheadrightarrow \langle x, t \mid t^{h_i}, x^{1-\alpha}t^{-j'}x^{-\beta}t^{j'}x^\alpha t^{-j'}x^\beta t^{j'} \rangle = E(h_i; \alpha, \beta; 0, j'),$$

where the second relator is cyclically reduced since $\alpha \neq 1$. This group is an extension of $G_{h_i}(x_0[x_0^\alpha, x_{j'}^\beta])$ which is infinite since $h_i \geqslant 4$ and each relator involves only two generators [12]. Similarly, since $\beta \neq -1$, if $h_j > 3$ then G is infinite.

This leaves the case $1 < h_i, h_j \leqslant 3$. Since G is irreducible it follows that $h_i \neq h_j$. Applying elementary moves if necessary, we can assume that $h_i = 2$ and $h_j = 3$, and since the conditions on α and β are symmetric we can assume that they are unchanged. Therefore there is an epimorphism from $E(n; \alpha, \beta; i, j)$ to $E(6; \alpha, \beta; 2, 3)$ which is an extension of $G_6(x_0[x_2^\alpha, x_3^\beta])$ which is proved infinite in [13]. $\square$

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